LAMINAR BOUNDARY LAYER IN A NON-NEWTONIAN FLUID. QUALITATIVE DISCUSSION

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1. In chemical engineering, flows of such non-Newtonian fluids as different kinds of pulp, suspensions, polymer mixtures, and solutions are widely used. The flow index n of such fluids can vary between zero and values of order ten. The equation for a self-modeling laminar boundary layer is often used in calculating such flows. The solutions of this equation are numerous and diverse, and depend crucially on the flow index n and on the boundary conditions. Hence, it is of interest to classify the solutions of the equation of a self-modeling laminar boundary layer for a non-Newtonian fluid when the flow index is varied from zero to infinity (in principle) on the basis of a qualitative study of the differential equations. The first qualitative study of the equation for a plane laminar boundary layer was carried out in [1] for a Newtonian fluid ($n = 1$). It was shown that when the surface of a plate is moved in the direction opposite to an external flow, the self-modeling boundary-layer equation either has no solution or has two solutions with different coefficients of friction, depending on the velocity of the surface of the plate. This result was obtained in [2, 3], independently of [1]. In [4, 5] the equation for a laminar boundary layer was considered for the flow of a non-Newtonian fluid with a flow index in the regions $2 > n > 1$ and $1 \geq n > 0.5$. The study of [4] was extended in [6] and spatial localization of the boundary layer for $2 > n > 1$ was reported.

In the present paper the self-modeling flow regime in a laminar boundary layer of a non-Newtonian fluid is studied in the general case, without the restriction to a certain region of positive values of n. The coordinates of the singular points, the characteristic numbers determining the types of these singular points, and the characteristics of the singular solutions are found as functions of the parameter n. This allows one to classify the solutions for values of n which have not yet been considered ($n \leq 1/2$, $n \geq 2$), and to refine, to a certain degree, the known results. We use the theory of continuous groups, the idea of a two-sheeted phase plane joined at infinity [4], and the Poincaré transformation. In order to analyze in detail the transition of the trajectories from one sheet of the phase plane to the other, we consider a sphere, made up of Poincaré hemispheres joined in a special way [7], where the hemispheres uniquely correspond to the two sheets.

2. We consider the equation for a self-modeling laminar boundary layer of a non-Newtonian fluid [8]

$$F = |f''|^{n-1}f''' + f''f = 0, \tag{2.1}$$

where the function $f(\eta)$ is related to the stream function $\psi(x, y)$ by the expression

$$\psi(x, y) = U_0 x\left(n\frac{n+1}{\mathrm{Re}_x}\right)^{\frac{1}{n+1}} f(\eta),$$

the self-modeling variable is given by

$$\eta = Ayx^{-\frac{1}{n+1}}, \quad A = \left[\frac{\rho U_0^{2-n}}{k(n+1)}\right]^{\frac{1}{n+1}},$$

the longitudinal and transverse velocities, and the tangential stress are given by the equations

$$u(x, y) = U_0 f'(\eta), \quad v(x, y) = \frac{U_0}{n+1}\left(\frac{n+1}{\mathrm{Re}_x}\right)^{\frac{1}{n-1}}[\eta f(\eta) - f(\eta)],$$
$$\mathrm{Re}_x = \rho x^n k^{-1} U_0^{2-n}, \quad \tau = k\,|\,\partial u/\partial y\,|^{n-1}\,\partial u/\partial y.$$

Equation (2.1) admits a one-parameter group of transformations [4]

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$$\eta^0 = a\eta, \quad f^0 = a^{\frac{1-2n}{2-n}} f, \tag{2.2}$$

and under these transformations the three-fold continued infinitesimal operator [9]

$$\underset{3}{X} = \xi \frac{\partial}{\partial \eta} + \varphi \frac{\partial}{\partial f} + \varphi \frac{\partial}{\partial f'} + \sigma \frac{\partial}{\partial f''} + R \frac{\partial}{\partial f'''}$$

satisfies the equation

$$\underset{3}{X} F = 0.$$

We find, using (2.2)

$$f^{0\prime} = a^{\frac{1+n}{n-2}} f', \quad f^{0\prime\prime} = a^{\frac{3}{n-2}} f''. \tag{2.3}$$

From (2.2) and (2.3) we have

$$\left(\frac{f^0}{f}\right)^{\frac{2-n}{1-2n}} = a, \quad \left(\frac{f^{0\prime}}{f'}\right)^{\frac{n-2}{1+n}} = a, \quad \left(\frac{f^{0\prime\prime}}{f''}\right)^{\frac{n-2}{3}} = a. \tag{2.4}$$

From the first and second equations of (2.4), and also the first and third, we find the following expressions for the invariants when $n \neq 0.5$

$$q = f' |f|^{\frac{1+n}{1-2n}}, \quad p = f'' |f|^{\frac{3}{1-2n}}; \tag{2.5}$$

with the help of these invariants (2.1) reduces to a pair of first-order equations

$$\frac{dp}{d\eta} = -|f|^{\frac{2-n}{2n-1}} p \left(|p|^{1-n} + \frac{3q}{2n-1}\right) \operatorname{sign} f; \tag{2.6}$$

$$\frac{dq}{d\eta} = |f|^{\frac{2-n}{2n-1}} \left(p - \frac{n+1}{2n-1} q^2 \operatorname{sign} f\right) \tag{2.7}$$

and to the two quadratures

$$f = \exp \int \frac{\operatorname{sign} f \,(2n-1)\, q\,dq}{(2n-1)\,p - (n+1)\,q^2 \operatorname{sign} f}; \tag{2.8}$$

$$\eta = \int q\,|f|^{\frac{n+1}{1-2n}}\, df. \tag{2.9}$$

The invariants p and q cannot be used when $n = 1/2$. In this case, from the second and third equations of (2.4), we obtain the invariant

$$w = f'' |f'|^{-2}, \tag{2.10}$$

and with the help of this invariant (2.1) can be reduced to a pair of first-order equations

$$\frac{dw}{d\eta} = -|f''|^{1/2} (fw + 2w^{3/2} \operatorname{sign} f'); \tag{2.11}$$

$$\frac{df}{d\eta} = |f''|^{1/2} \operatorname{sign} f' \,|w|^{-1/2}. \tag{2.12}$$

The second variable of the system (2.11) and (2.12) is f, since according to (2.3), $f^0 = f$ when $n = 1/2$.

3. The solution of (2.1) for $n \neq 1/2$ [with the help of (2.6) and (2.7)] can be represented by trajectories in a two-sheeted phase plane (p, q) where on one sheet $f > 0$ and on the other $f < 0$ [4]. We consider the singular points of the system (2.6) and (2.7) located within the finite part of the phase plane. Equating the right-hand sides of (2.6) and (2.7) to zero, we find that on each sheet of the phase plane, for $n \neq 1/2$, there exist two singular points with finite values of the coordinates. On the plane $f > 0$ the singular points are α, with the coordinates

$$p_0 = \delta\left(|2n-1|\frac{n+1}{9}\right)^{\frac{1}{2n-1}}, \quad q_0 = \frac{1-2n}{3}|p_0|^{1-n}, \quad \delta = \begin{cases} +1, & n > 1/2, \\ -1, & n < 1/2 \end{cases} \tag{3.1}$$

and $\beta(0, 0)$, and on the plane $f < 0$ the points are $\alpha'(-p_0, q_0)$ and $\beta'(0, 0)$.

Linearizing (2.6) and (2.7) in the neighborhoods of the points α and α', we obtain

$$\frac{dp}{dq} = \frac{\operatorname{sign} f\,(n-1)\,|p_0|^{1-n}\,(p-(\pm p_0)) - \operatorname{sign} f\,\frac{3(\pm p_0)}{2n-1}(q-q_0)}{p-(\pm p_0)+2\frac{n+1}{3}\operatorname{sign} f\,|p_0|^{1-n}(q-q_0)}. \tag{3.2}$$

The characteristic numbers are found from the discriminant of (3.2)

$$\lambda_{1,2} = \operatorname{sign} f\,|p_0|^{1-n}\frac{n5-1}{6} \pm |p_0|^{1-n}\left[\frac{(5n-1)^2}{36} - \frac{n+1}{3}(2n-1)\right]^{1/2}. \tag{3.3}$$

It follows from (3.3) that the singular points α and α' are saddle points when $n < 1/2$, focal points when $0.6 < n < 21.4$, and nodes when $0.6 > n > 0.5$. The focal and nodal points are stable on the plane $f < 0$ and unstable on the plane $f > 0$. The points α and α' describe the solutions

$$f_{\alpha,\alpha'} = \pm\left\{\pm\frac{2-n}{3}\left(|2n-1|\frac{1+n}{9}\right)^{\frac{1-n}{2n-1}}\eta \pm \frac{2-n}{1-2n}A\right\}^{\frac{1-2n}{2-n}}$$

for $n \neq 2$. For $n = 2$ they describe the solutions

$$f_{\alpha,\alpha'} = e^{\pm(-\eta+A)}.$$

The plus sign in these solutions corresponds to the point α, and the minus sign to α'.

The singular points β and β' are complicated, and, as follows from constructions along the zero isoclines, they are of the saddle-point–nodal-point type. The singular points α and α' ($n \neq 1/2$) and β and β' ($n > 1/2$) are equilibrium positions of the system, and at these points the derivatives $p' = q' = 0$. For $n < 1/2$ the trajectories on the plane $f > 0$ converge toward the point β with the derivative $f' < 0$ ($q < 0$), and diverge from the point β' on the plane $f < 0$, also with the derivative $f' < 0$. Since $q = 0$ at the points β and β', we obtain that $f = 0$ at these points and the derivatives p' and q' are indeterminant (0/0). This means that the points β, β' cannot be positions of equilibrium when $n < 1/2$. The point representing the state of the system does not approach β asymptotically, but passes through β, β' in going from one sheet of the phase plane ($f > 0$) to the other ($f < 0$).

We consider which of the solutions of the system (2.6) and (2.7) are singular. We divide (2.6) by (2.7) ($n \neq 1/2$)

$$\frac{dp}{dq} = \frac{p|p|^{1-n} + \frac{3}{2n-1}qp}{-p\operatorname{sign} f + \frac{n+1}{2n-1}q^2} = R_0(p, q). \tag{3.4}$$

The singular solutions are those curves in the (p, q) plane which are solutions of (3.4) and along which the derivative $\partial R_0/\partial p = \infty$ [10]. Differentiating R_0, we obtain

$$\frac{\partial R_0}{\partial p} = \frac{\left[\frac{3q}{2n-1}+(2-n)|p|^{1-n}\right]\left[q^2+\frac{n+1}{2n-1}-p\operatorname{sign} f\right]+p\left[\frac{3q}{2n-1}+|p|^{1-n}\right]\operatorname{sign} f}{\left(q^2\frac{n+1}{2n-1}-p\operatorname{sign} f\right)^2}. \tag{3.5}$$

It follows from (3.4) and (3.5) that the singular solution is the straight line $p = 0$, and only for $2 > n > 1$.

In the case $n = 1/2$ the solution of (2.1) can be represented by the trajectories of the system (2.11) and (2.12) in the two-sheeted (w, f) phase plane. The two sheets correspond to positive and negative values of the derivative f'. It follows from (2.11) and (2.12) that in the finite part of the (w, f) phase plane (for $f' > 0$ and for $f' < 0$) there are no singular points. There are also no singular solutions.

4. We consider the singular points of the system of equations (2.6) and (2.7) which lie at infinity. In order to do this, we perform a first Poincaré transformation [7]

$$q = 1/z,\ p = u_0/z. \tag{4.1}$$

In terms of the variables u and z, the Eqs. (2.6) and (2.7) take the form

$$\frac{du_0}{d\eta} = |f|^{\frac{2-n}{2n-1}} \frac{u_0}{z}\left(-u_0 z + \frac{n-2}{2n-1}\operatorname{sign} f - \operatorname{sign} f\,|u_0|^{1-n}\operatorname{sign} z\,|z|^n\right); \tag{4.2}$$

$$\frac{dz}{d\eta} = -|f|^{\frac{2-n}{2n-1}}\left(u_0 z - \frac{n+1}{2n-1}\operatorname{sign} f\right). \tag{4.3}$$

The point $z = 0$, $u_0 = 0$ is a singular point of the system (4.2) and (4.3). It corresponds to two points B and B' of the Poincaré sphere. These points are the intersections of the equator $z = 0$ and the axis Q, which goes outward from the center of the sphere parallel to the q axis (see [7]). In the neighborhood of the singular point

$$\frac{du_0}{dz} = \frac{n-2}{n+1}\frac{u_0}{z},$$

and therefore the characteristic numbers are $\lambda_1 = n - 2$ and $\lambda_2 = n + 1$. This singular point is a node for $n > 2$ and a saddle point for $n < 2$ and the equator is a solution of the system (4.2) and (4.3). When $n = 2$ the system (4.2) and (4.3) does not have singular points on the equator, and the equator is not a solution of these equations.

We study the intersection points of the equator with the P axis, which goes out from the center of the sphere parallel to the p axis (points D, D'). We apply a second Poincaré transformation

$$q = v_0/z,\ p = 1/z. \tag{4.4}$$

In terms of the variables v_0 and z, Eqs. (2.6) and (2.7) can be written as

$$\frac{dv_0}{d\eta} = |f|^{\frac{2-n}{2n-1}} \frac{1}{z}\left(z + \frac{2-n}{2n-1} v_0^2 \operatorname{sign} f + \operatorname{sign} z \operatorname{sign} f v_0 |z|^n\right); \tag{4.5}$$

$$\frac{dz}{d\eta} = |f|^{\frac{2-n}{2n-1}}\left(\operatorname{sign} z\,|z|^n + \frac{3}{2n-1} v_0\right)\operatorname{sign} f. \tag{4.6}$$

The point $v_0 = 0$, $z = 0$ is a singular point of (4.5) and (4.6) for $n > 1$. In the neighborhood of this point, neglecting (for $n > 1$) in the numerator of the right-hand side of (4.5) terms higher than second order, and in the right-hand side of (4.6) terms higher than first order, we obtain

$$\frac{dv_0}{dz} = \frac{z + \frac{2-n}{2n-1} v_0^2 \operatorname{sign} f}{\frac{3}{2n-1} z v_0 \operatorname{sign} f}. \tag{4.7}$$

After performing the substitution $\nu = z v_0^{-2}$ in (4.7), we find that it has the integral

$$|v_0^2 - 2\operatorname{sign} fz| = c|z|^{\frac{2(2-n)}{3}} \tag{4.8}$$

[c is a constant of integration ($c > 0$)].

It can be shown that (4.8) has two solutions in each of the regions $f > 0$ and $f < 0$. We consider the case $f > 0$. Then

$$|v_0^2 - 2\operatorname{sign} fz| = |v_0^2 - 2z| = \begin{cases} \left| |v_0^2 - 2|z|| \right|, & z > 0, \\ \left| |v_0^2 + 2|z|| \right|, & z < 0. \end{cases}$$

The first expression inside the curly brackets has two possible values:

$$|v_0^2 - 2|z|| = \begin{cases} v_0^2 - 2|z|, & v_0^2 > 2z, \\ 2|z| - v_0^2, & v_0^2 < 2z. \end{cases}$$

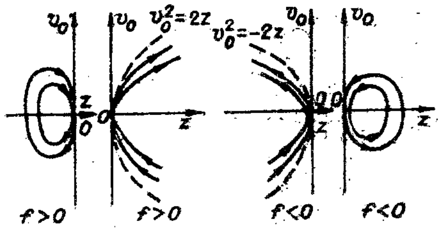

Fig. 1

And thus, when $z < 0$

$$v_0 = \pm\left(c\,|z|^{\frac{2(2-n)}{3}} - 2\,|z|\right)^{1/2}. \tag{4.9}$$

When $z > 0$ there are two possible solutions:

$$v_0 = \pm\left(c\,|z|^{\frac{2(2-n)}{3}} + 2\,|z|\right)^{1/2} \quad \text{for} \quad v_0^2 > 2z; \tag{4.10}$$

$$v_0 = \pm\left(2\,|z| - c\,|z|^{\frac{2(2-n)}{3}}\right)^{1/2} \quad \text{for} \quad v_0^2 < 2z. \tag{4.11}$$

However, the solution (4.11) does not occur in the real part of the complex plane, since when $n > 1$ the expression under the radical in (4.11) is negative. Hence there exist two solutions in the region $f > 0$: (4.9) for $z < 0$ and (4.10) for $z > 0$, and $v_0^2 > 2z$ when $z > 0$. In a similar way it can be shown that in the region $f < 0$ there are two solutions: (4.10) for $z < 0$ and (4.9) for $z > 0$, and $|z| < v_0^2/2$ when $z < 0$.* Graphs of the functions (4.9) and (4.10) in the regions $f > 0$ and $f < 0$ are shown in Fig. 1 for $z > 0$ and $z < 0$. The distribution of trajectories in the neighborhood of the point $v_0 = 0$, $z = 0$ for $1/2 < n \leq 1$ has the same form and can be determined by constructing the zero isoclines.

We consider the infinitely distant points of the (w, f) plane. At these points $f = \pm\infty$ and so there are no transitions of trajectories from one sheet of the phase plane to the other (as follows from Sec. 3, there are no singular points in the finite portion of the (w, f) phase plane).

We introduce a first Poincaré transformation $w = u_1/z_1$, $f = 1/z_1$. Then the system (2.11) and (2.12) reduces to the form

$$\frac{du_1}{d\eta} = -|f''|^{1/2}\frac{1}{z_1}\left(\operatorname{sign} f' z_1^2 |z_1|^{1/2} \operatorname{sign} u_1 |u_1|^{1/2} + u_1 + u_1 |u_1|^{3/2} \operatorname{sign} z_1 |z_1|^{1/2} \operatorname{sign} f'\right); \tag{4.12}$$

$$\frac{dz_1}{d\eta} = -z_1^2 |f''|^{1/2} \operatorname{sign} f' \,|z_1 u_1^{-1}|^{1/2}. \tag{4.13}$$

It follows from (4.12) and (4.13) that the point $z_1 = 0$, $u_1 = 0$ is singular. Dividing (4.12) by (4.13) and neglecting in the numerator of the right-hand side the ratios of terms of higher order than 3/2, we find the following equation in the neighborhood of the point $z_1 = 0$, $u_1 = 0$

$$\frac{du_1}{dz_1} = \frac{u_1 |u_1|^{1/2}}{z_1^3 |z_1|^{1/2}} \operatorname{sign} f', \tag{4.14}$$

which has the integral

$$u_1 = \left(\frac{1}{5} \operatorname{sign} f' |z_1|^{5/2} + K\right)^{-2} \tag{4.15}$$

(K is a constant of integration). From (4.14) and (4.15) it can be shown that in the neighborhood of the singular point $z_1 = 0$, $u_1 = 0$, the distribution of trajectories forms a node.

*We note that the authors of [6] apparently did not recognize that the left-hand side of (4.8) is an absolute value, and therefore of the two solutions (4.9) and (4.10), they considered only (4.10).

We introduce a second Poincaré transformation $w = 1/z_1$, $f = v_1/z_1$. Then (2.11) and (2.12) transform to

$$\frac{dv_1}{d\eta} = |f''|^{1/2} \frac{1}{z_1} (\operatorname{sign} f' z_1^2 |z_1|^{1/2} + v_1^2 + 2v_1 |z_1|^{1/2} \operatorname{sign} f'); \tag{4.16}$$

$$\frac{dz_1}{d\eta} = |f''|^{1/2} (v_1 + |z_1|^{1/2} \operatorname{sign} f'). \tag{4.17}$$

The point $z_1 = 0$, $v_1 = 0$ is a singular point of the system (4.16) and (4.17). Keeping only the lowest-order terms on the right-hand sides of (4.16) and (4.17), we obtain, in the neighborhood of the point $z_1 = 0$, $v_1 = 0$:

$$\frac{dv_1}{dz_1} = 2 \frac{v_1}{z_1}.$$

From (4.18) the characteristic numbers at the singular point are $\lambda_1 = 2$, $\lambda_2 = 1$ and hence the singular point $z_1 = 0$, $v_1 = 0$ is a node.

5. The discussion of the invariants of (2.1) and its singular points in Secs. 2 through 4 has shown that there are six intervals of the index n with qualitatively different phase diagrams: $n > 2$, $n = 2$, $2 > n > 1$, $1 \geq n > 1/2$, $n = 1/2$, $n < 1/2$. A set of trajectories is constructed in the phase plane by considering the behavior of the trajectories in the neighborhoods of the singular points and the intersections of the trajectories with the zero isoclines. The question of a transition of the trajectories from one sheet of the phase plane to the other through a singular point is resolved by starting from the continuity of the function f and its derivatives across the transition, and the correspondence between the solutions and possible flows in the boundary layer. Equation (2.1) is invariant with respect to the reflection $(f, \eta) \to (-f, -\eta)$. Therefore each trajectory in the set of trajectories obtained for a given n can be correlated with the trajectory related to it by a reflection. Hence the set of trajectories for each value of n can be divided into two sets whose trajectories are related by this transformation.

Mappings of both sheets of the phase planes (for the above intervals in n) into circles (obtained by means of a previous mapping of the planes onto the lower Poincaré hemisphere) are shown in Fig. 2, where the origins of the coordinate axes p, q and w, f are taken at the centers of the circles (a through d, f, and e), respectively). The axes p and w are oriented upward, and q and f point to the right. The shaded circles denote the singular points through which the trajectories pass from the circle $f > 0$ ($f < 0$) to the circle $f < 0$ ($f > 0$), while the open circles denote the singular points corresponding to equilibrium positions. The trajectories belonging to the (second) set, obtained after the transformation $(f, \eta) \to (-f, -\eta)$ are denoted by putting primes on the letters, which correspond to a definite trajectory type. For greater clarity in the passing of the trajectories through the transition points, parts of the trajectories are crossed by primes, with an increase in the number of primes corresponding to the sequence of passages of the trajectories along the η coordinate. For all n except $n = 2$ the outer circumference of the circle is a trajectory of the system. For $n = 2$ each point of the circumference is a transition point (but not a singular point). Using the system of trajectories, the expressions for the invariants (2.5) and (2.10), and the defining equation (2.1), one can qualitatively construct for each value of n a system of integral curves $f(\eta)$.

Figure 3 shows the integral curves for different types of trajectories belonging to the first set only, since with the help of a reflection one can obtain the integral curves of the second set with no difficulty. It is significant that in spite of the different paths of the trajectories for different values of n, the corresponding integral curves are qualitatively close to one another. Hence they can be denoted by a single symbol. For all n (except $n = 1/2$) there exists the set of integral curves (and trajectories) b, S, c, S_0, a, r, which will be called the fundamental set (Figs. 2 and 3). This fundamental set, as shown in [1] for the boundary layer of a Newtonian fluid, corresponds to two different solutions with different friction coefficients in the flow of a fluid along a plate with a negative velocity of the surface of the plate. Hence for the flow of a non-Newtonian fluid with any $n \neq 1/2$, and in particular for the flow of a non-Newtonian fluid ($n \neq 1/2$) along a plate with a surface moving in the direction opposite to the external flow, there exist two self-modeling solutions with different coefficients of friction. The critical velocity of the surface of the plate, at which the steady self-modeling solution ceases to exist, depends weakly on the flow index, increasing in absolute value with increasing flow index [5]. If $n > 1/2$, then in addition to the fundamental set of solutions, there exist the solutions α, β, S_1 (Figs. 2a-d; 3b) and for $1 < n < 2$, there are c_1, c_2, formed by joining the solutions c, r, r' (Figs. 2c; 3c), and their reflections. For $n < 1/2$ there exist the solutions l_0, l_1, l_{11}, l_{12}, l_{1S}, α and their reflections (Figs. 2f, 3d), in addition to the fundamental set of solutions. It should be noted that the solutions α and α', denoted by the same letter as the equilibrium position, differ depending on n. In particular, $f_\alpha(\eta)$ for $1/2 < n < 2$ has a vertical asymptote,

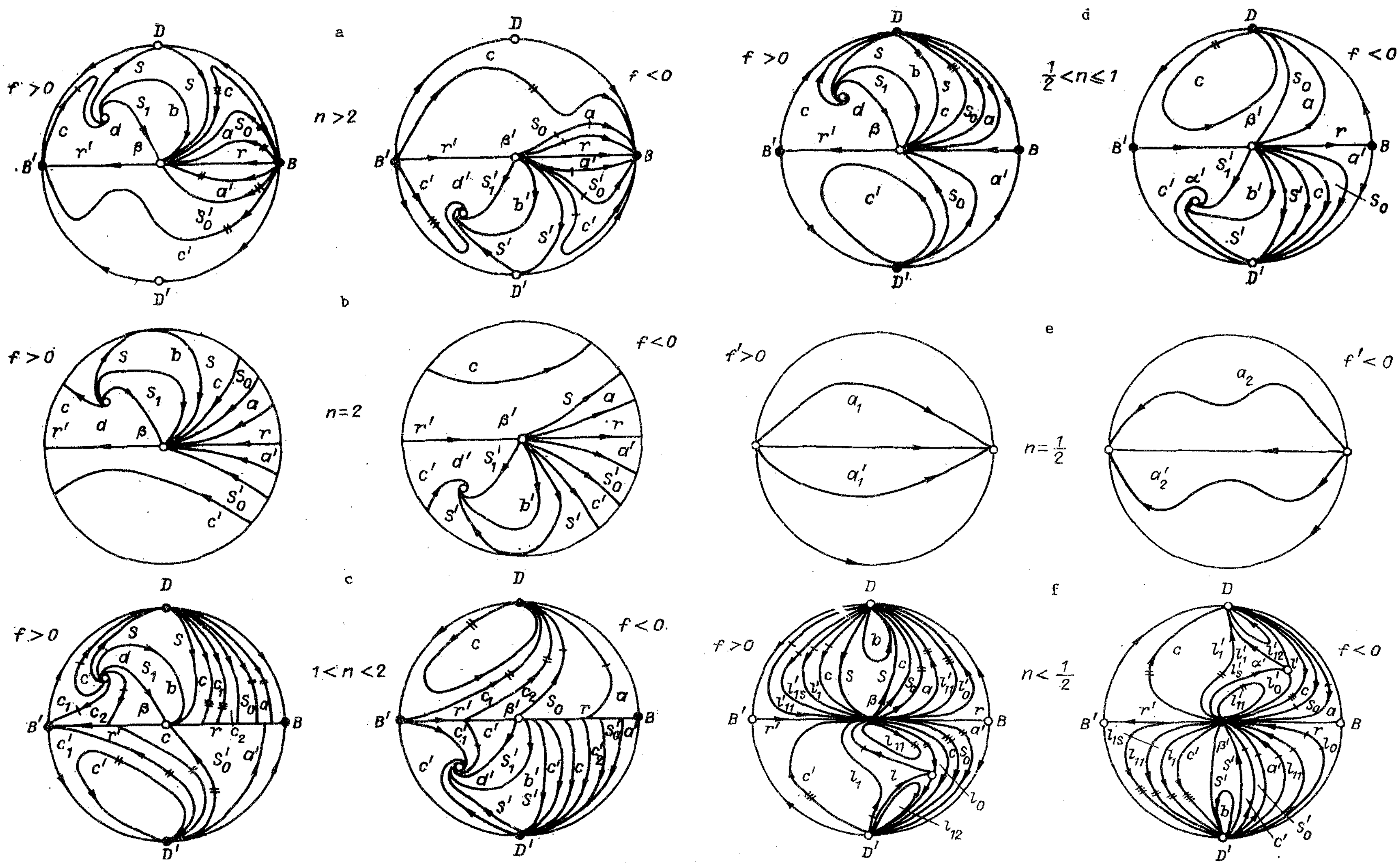

Fig. 2

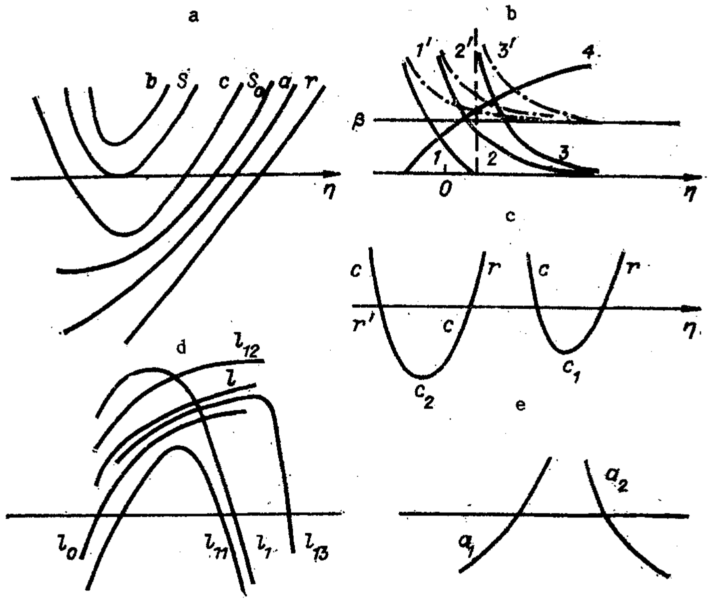

Fig. 3

while for $n \geq 2$ the solutions monotonically increase with decreasing η (curves 1-4 of Fig. 3b for $n > 2$, $n = 2$, $1/2 < n < 2$, $n < 1/2$). This means that the solutions b, S, c, S_1, which approach α with decreasing η, have different asymptotic behaviors of their left branches for $1/2 < n < 2$ and $n \geq 2$. As an example, in Fig. 3b the curves 1'-3' show the solution S_1 for $n > 2$, $n = 2$, $1/2 < n < 2$. The simplest set of trajectories occurs in the degenerate case when $n = 1/2$. Then the invariants p, q become irreducible (Fig. 2e). The corresponding integral curves are shown in Fig. 3e.

The integral curves of Fig. 3, depending on their shape and on the position of the origin of the coordinate η (in the upper or lower half-plane, on the increasing or decreasing branch, and so on) describe different flows in the boundary layer: a stationary plate (S) with injection (c_1, c, c_2) and exhaust (b, l_1, l_{11}, l_{1S}), a plate moving in the same direction as the external flow (c, S_0, a, c_1, c_2, l_{11}, l_1, l_{1S}, a_1, a_2) with exhaust (b, S, c, S_0, a, c_1, c_2, a_1, a_2, l_{11}, l, l_1, l_{1S}, l_0) and injection (l_{11}, l_1, l_{1S}, a_2, a_1, c, c_1, c_2, l_0, S_0, a), a plate moving in the opposite direction to the external flow (c, c_1, c_2, l_{11}) with exhaust (c, c_{10}, c_2, S, b, l_{11}, l_1, l_{1S}) and injection (c, c_1, c_2, l_{11}), a plate moving in a medium at rest (α for $1/2 < n \leq 2$, S_1, l_{12}), a plate moving with the same velocity as the external flow with injection and exhaust (r), a mixing layer of flows with zero (S_0, a_1, a_2) and finite association (a, a_1, a_2), a mixing layer where the flows are in opposite directions (c_2 when the upper joining points of the solutions r' and c go to infinity).

The solutions c_1 and c_2 are possible for $1 < n < 2$ and describe the phenomenon of localization of the boundary layer, which was observed in [6] for the flow of a dilating fluid in this interval of n.

It was asserted in [11] that the boundary layer is also localized for $n > 2$. However, as follows from Sec. 3, the solutions r and r' are not singular in this case. Therefore solution c cannot be joined with the solutions r and r' for finite values of the variable η and localization of the boundary layer cannot occur for $n > 2$.*

It is important to note that for an equation of the type considered here (involving the absolute value of a function which can change sign) it is necessary in constructing the phase diagrams to consider two Poincaré spheres, corresponding to the two sheets of the phase plane. The transition from one sheet to the other corresponds to the transition of the trajectories from one Poincaré sphere to the other through the transitional singular points. It is of interest to introduce a manifold which would be unique for positive and negative values of the function f. We consider flows with n greater than one-half. In this case the transitional singular points lie on the equator of the Poincaré sphere. If we join two Poincaré hemispheres along the equator (let the upper hemisphere correspond to $f < 0$, and the lower one to $f > 0$), then when f changes sign the trajectories go from one hemisphere to the other and belong to a single manifold.

*The incorrectness of the assertion in [11] on the localization of the boundary layer for $n > 2$ results from the fact that the authors of [11] considered the behavior of the solutions in the limit as f'' goes to zero. However (2.1) was not used, but an equation obtained by transforming it with a transformation which is incorrect for $n > 2$ and $f'' \to 0$.

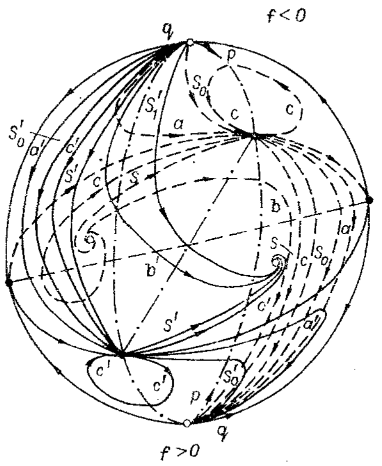

Fig. 4

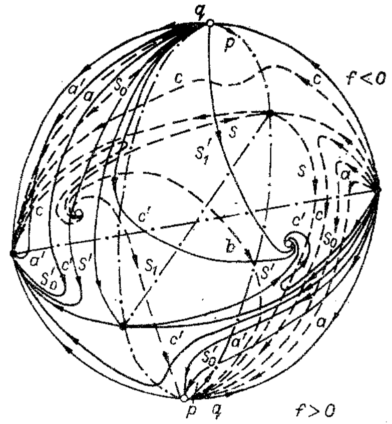

Fig. 5

The analysis shows that for the set of necessary conditions of the transition indicated in the beginning of Sec. 5, it is sufficient to impose a reflection on the upper hemisphere (up to its joining with the lower hemisphere) with respect to a plane passing through the points B and B', and perpendicular to the equator. In this manifold the double-valued nature of the correspondence between the points of the Poincaré sphere and the mapped planes does not appear, and the set of solutions completely reduces to the solutions represented in Fig. 2. The manifolds for flow indices $1/2 < n \leq 1$ and $n > 2$ are shown in Figs. 4 and 5.

When $n < 1/2$ the transition from one sheet of the phase plane to the other goes through the origin of the coordinate system. In order to have a smooth transition of the trajectories on the sphere through this point it is necessary to nest the lower Poincaré hemisphere (corresponding to $f > 0$) inside the lower Poincaré hemisphere corresponding to $f < 0$, after rotating it by 180°.

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